Wave Mechanics Tutorial Session

Cracking the Electron Identity

R I S H I K E S H   V A I D Y A

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April 22, 2007

Resolving Wave-Particle Dualism of Electron:

When do we say something has a particle nature ?

Exp 1: An Experiment with Bullets

Resolving Wave-Particle Dualism of Electron:

Imagine a Drunkard spraying bullets with a Gun randomly in all directions

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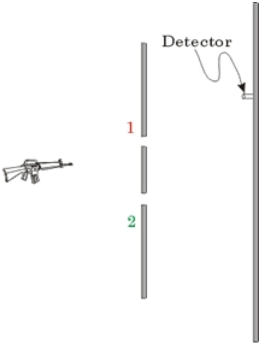
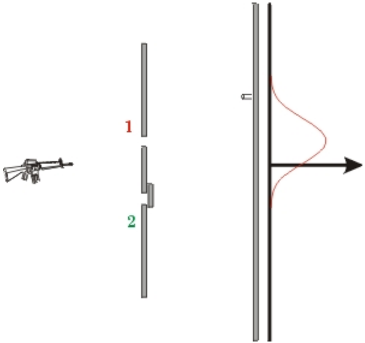


Resolving Wave-Particle Dualism of Electron:

Imagine a Drunkard spraying bullets with a Gun randomly in all directions

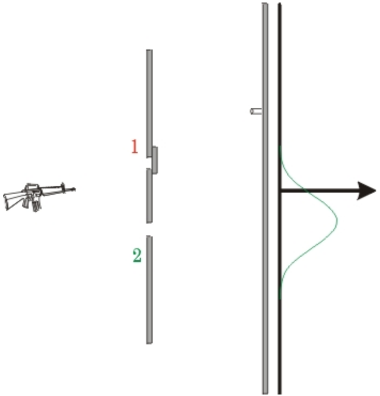
Slit 1 open:
What we can measure is the probability P_1 of finding a bullet a distance x from the center.



Resolving Wave-Particle Dualism of Electron:

Imagine a Drunkard spraying bullets with a Gun randomly in all directions

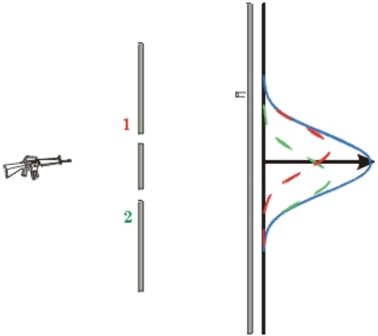
Slit 2 open:
Now we measure the probability P_2 of finding a bullet a distance x from the center.



Resolving Wave-Particle Dualism of Electron:

Imagine a Drunkard spraying bullets with a Gun randomly in all directions

both the slits open:
We measure the probability $P_{12} = P_1 + P_2$ of finding a bullet a distance x from the center.



Resolving Wave-Particle Dualism of Electron:

Imagine a Drunkard spraying bullets with a Gun randomly in all directions

Definition of Particle Nature

A physical entity has a particle nature if its probability distribution adds in the manner of bullets when it is subjected to two slit experiment.

Ripple Tank Experiment to Demonstrate Wave Nature

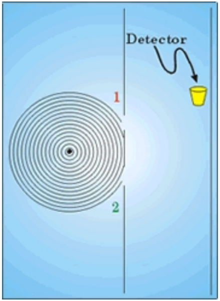
When do we say that something has a wave nature ?

Exp 2: An Experiment with ripples in water tank

Ripple Tank Experiment to Demonstrate Wave Nature

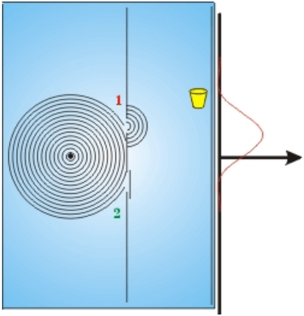
Consider tapping the surface of a water filled tank and just as before we have two slits

Ripple Tank Experiment to Demonstrate Wave Nature



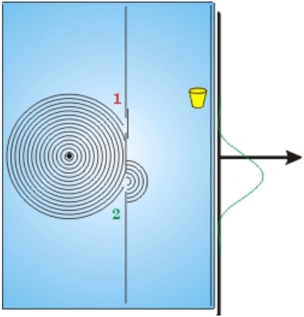
Ripple Tank Experiment to Demonstrate Wave Nature

Slit 1 open:
We can measure the Intensity distribution of wave energy arriving at the backstop $I_1 = |h_1|^2$ at any distance x from the center.



Ripple Tank Experiment to Demonstrate Wave Nature

Slit 2 open:
Now we measure the intensity $I_2 = |h|^2$ at any distance x from the center.



Ripple Tank Experiment to Demonstrate Wave Nature

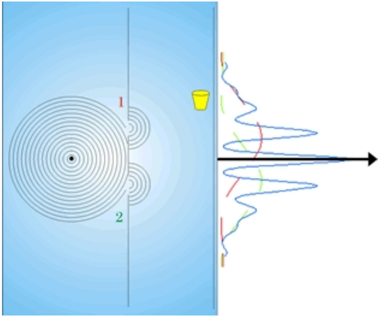
Both the slits open:

We measure the intensity

$I_{12} = |h_1 + h_2|^2$

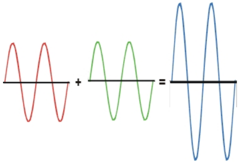
$I_{12} = |h_1|^2 + |h_2|^2$
$+2|h_1||h_2|cos\delta$

at a distance x from the center. Notice the interference pattern – it is the signature of wave nature



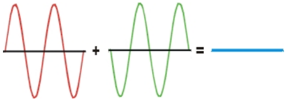
Constructive Interference

Waves are in phase



Destructive Interference

Waves are out of phase



Ripple Tank Experiment to Demonstrate Wave Nature

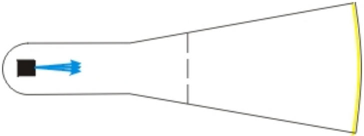
Definition of wave nature

A physical entity has a wave nature if its intensity distribution shows interference pattern when it is subjected to two slit experiment.

Cracking the Identity of Electrons:
Two Slit Experiment with Electrons

Consider a thought experiment with electrons fired from an electron gun and passing through two slits as before

Every TV houses an Electron Gun



"Everybody knows" electrons are particles. They after all have mass $\sim 10^{-30} Kg$. Lets crack the identity once and for all.

Are the Electrons Conspiring Against Us ?

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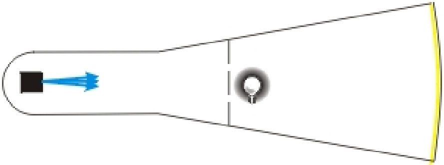
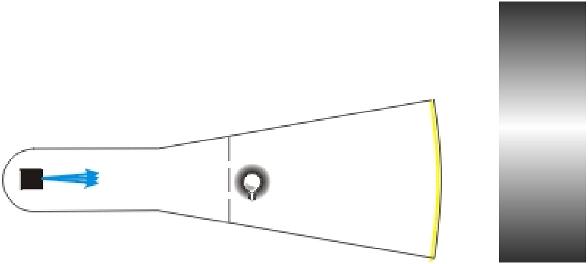
- **Interpretation 1:** May be the electrons passing through the two slits are conspiring to produce the interference pattern. This does not seem to be the case as even when you allow one electron at a time, after waiting for sufficiently long time, they still produce the interference pattern
- **Employ Detectives:** Track their path and observe their shady behaviour.

Are the Electrons Conspiring Against Us ?



Are the Electrons Conspiring Against Us ?



The Quantum Wonderland lies beyond the boundaries of objective reality, certainty and Determinism. It redefines the task of Physics. The business of Physics is not to describe how nature is but what we can say about nature.

So How to Solve a Quantum System ?

- In Newtonian Mechanics you can determine time evolution of any physical system if you know initial position and momentum as boundary conditions.
- In Quantum Mechanics (QM), you cannot determine position momentum simultaneously due to Heisenberg's Uncertainty principle.
  - Spread in the values of physical observables like position, momentum etc., is inherent in QM and all you can know is the probability for the system to have a certain value for any physical observable.
  - Quantum mechanical entities like electron are essentially described by waves of probability given by ψ (just as classical entities are described by position and momentum).

What is a well-behaved wavefunction ψ ?

- In general a wave equation has many possible solutions but not all of them correspond to physically admissible solutions.
- Remember – in general ψ is a complex quantity and hence cannot correspond to real physical situation
- $\psi^*\psi = |\psi|^2$ is however real and gives the probability density.
- Now total probability can range from 0 to 1. So if particle exists somewhere in space:

  $\boxed{\int_{\text{all space}} \psi^*\psi d\tau = 1}$ **Normalization condition**
- ψ must also be single valued. Momentum considerations require that $\partial\psi/\partial x, \partial\psi/\partial y, \partial\psi/\partial z$, be finite continuous and single valued.

ψ is obtained by solving Schroedinger equation

- Schroedinger equation is to Quantum mechanics what Newton's law is to classical mechanics
- Time dependent Schroedinger equation in one-dimension –

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

- When potential $V(x)$ does not depend on time, we can write –

$$\Psi(x,t) = \psi(x)e^{-i\omega t}$$

where $\psi(x)$ is obtained by solving time-independent Schroedinger eq.

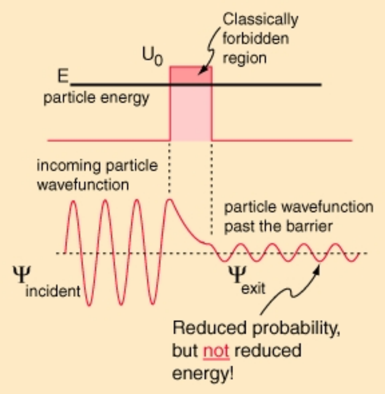
$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

Quantum Magic: Tunneling through the barrier

If a mountain falls on your way the only way ahead is to climb it. You cannot tunnel through it.

Quantum Magic: Tunneling through the barrier

However a quantum particle can actually tunnel through a barrier without digging the barrier



Problem: Show that the approximate probability of particle to tunnel through the barrier of length L is e^{-2kL}.

The Schroedinger eq. in the region potential free regions I and III takes the form:

$$\begin{aligned}\frac{\partial^2\psi_I}{\partial x^2}+\frac{2m}{\hbar^2}E\psi_I &= 0\\ \frac{\partial^2\psi_{III}}{\partial x^2}+\frac{2m}{\hbar^2}E\psi_{III} &= 0\end{aligned}$$

The solutions are given as:

$$\begin{aligned}\psi_I &= \psi_{I+}+\psi_{I-}=Ae^{ik_1x}+Be^{-ik_1x}\\ \psi_{III} &= \psi_{III+}+\psi_{III-}=Fe^{ik_1x}+Ge^{-ik_1x}\end{aligned}$$

where $k_1=\frac{\sqrt{2mE}}{\hbar}$ is the wave number of de-Broglie waves.

If v is the group velocity of incident wave, then

$$S = v|\psi_{I+}|^2$$

is the flux of paticles that arrive at the barrier (that is number of particles per unit time per unit area that arrive there). Also we note that $G = 0$.

The transmission probability T for a particle to pass through the barrier is the ratio

$$T = \frac{|\psi_{III+}|^2 v}{|\psi_{I+}|^2 v} = \frac{FF^*}{AA^*}$$

That means, T is the fraction of incident particles that succed in tunneling through the barrier. Classically T $=$0 because a particle with $E < 0$ cannot exist inside the barrier.

Let us see what happens quantum mechanically.

Region II: Solutions inside the barrier

In region II Schroedinger eq. is given as:

$$\frac{\partial^2 \psi_{II}}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi_{II} = 0$$

and its solution is –

$\psi_{II} = \psi_{II+} + \psi_{II-} = Ce^{ik'x} + Be^{-ik'x}$ where $k' = \frac{\sqrt{2m(E-V)}}{\hbar}$.

Since $E < V$ k' is imaginary and it is convenient to define k_2 as

$$k_2 = -ik' = \frac{\sqrt{2m(V - E)}}{\hbar}$$

In terms of k_2 then $\psi_I I$ is written as

$$\psi_{II} = \psi_{II+} + \psi_{II-} = Ce^{-k_2 x} + Be^{k_2 x}$$

Well behaved ψ respects boundary conditions

At $x = 0$

$$\begin{aligned} \psi_I &= \psi_{II} \\ \frac{\partial \psi_I}{\partial x} &= \frac{\partial \psi_{II}}{\partial x} \end{aligned}$$

At $x = L$

$$\begin{aligned} \psi_{II} &= \psi_{III} \\ \frac{\partial \psi_{II}}{\partial x} &= \frac{\partial \psi_{III}}{\partial x} \end{aligned}$$

$$\begin{aligned} A + B &= C + D \\ ik_1 A - ik_1 B &= -k_2 C + k_2 D \\ Ce^{-k_2 L} + De^{k_2 L} &= Fe^{ik_1 L} \\ -k_2 Ce^{-k_2 L} + k_2 De^{k_2 L} &= ik_1 Fe^{ik_1 L} \end{aligned}$$

Approximations

The above eq. may be soloved for (A/F) to give–
$\left(\frac{A}{F}\right) = \left[\frac{1}{2} + \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{i(k_1+k_2)L} + \left[\frac{1}{2} - \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{i(k_1+k_2)L}$
Assume that V is much greater than E. Then $k_2/k_1 >> k_1/k_2$.

Then
$\frac{k_2}{k_1} - \frac{k_1}{k_2} \sim \frac{k_2}{k_1}$
Let us also assume that barrier is wide enough for ψ_{II} to be severly weakened between $x = 0$ and $x = L$. This means that $k_2 L >> 1$ and $e^{k_2 L} >> e^{-k_2 L}$. Hence

$$\left(\frac{A}{F}\right) = \left(\frac{1}{2} + \frac{ik_2}{4k_1}\right) e^{(ik_1+k_2)L}$$

$$\frac{AA^*}{FF*} = \left(\frac{1}{4} + \frac{k_2^2}{16k_1^2}\right) e^{2k_2 L}$$

Transmission Probabilty

$$T = \frac{FF^*}{AA^*} = \left[\frac{16}{4 + (k_2/k_1)^2}\right] e^{-2k_2 L}$$

From the definition of k_1 and k_2 we see that

$$\left(\frac{k_2}{k_1}\right)^2 = \frac{V}{E} - 1$$

This means that quantity in square bracket varies much less with E and V then does the exponential. To a reasonable approximation, we can write

$$T = e^{-2k_2 L}$$

credit for figures: http://www.upscale.utoronto.ca/GeneralInterest/Harrison/DoubleSlit/DoubleSlit.html