Art Gallery Theorems and Approximation algorithms

Subir Kumar Ghosh
School of Technology & Computer Science
Tata Institute of Fundamental Research, Mumbai 400005
http://www.tcs.tifr.res.in/∼ghosh/

The art gallery problem is to determine the number of guards that are sufficient to *cover* or *see* every point in the interior of an art gallery. An art gallery can be viewed as a polygon P with or without holes with a total of n vertices and guards as points in P. Any point $z \in P$ is said to be visible from a guard g if the line segment joining z and g does not intersect the exterior of P. Usually the guards may be placed anywhere inside P. If the guards are restricted to vertices of P, we call them *vertex guards*. If there is no restriction, the guards are referred as *point guards*. Point and vertex guards are also referred as *stationary guards*. If the guards are mobile, i.e., able to patrol along a segment inside P, they are called *mobile guards*. If the mobile guards are restricted to edges of P, they are called *edge guards*.

The art gallery problem was first posed by Victor Klee for stationary guards. Chavatal [4] proved that a simple polygon P needs at most $\lfloor n/3 \rfloor$ stationary guards. Fisk [6] later gave a simple proof of this result using coloring technique. O'Rourke [13] showed that P needs at most $\lfloor n/4 \rfloor$ mobile guards. For a simple orthogonal polygon P, i.e., the edges of P are horizontal or vertical, Kahn et al. [11] proved that P needs at most $\lfloor n/4 \rfloor$ stationary guards. Aggarwal [1] showed that P needs at most $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards. This bound also holds for edge guards as shown by Bjorling-Sachs [2].

For a polygon P with h holes, O'Rourke [13] showed that P needs at most $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards. Hoffmann et al. [10] and Bjorling-Sachs and Souvaine [3] proved independently that P can always be guarded with $\lceil \frac{n+h}{3} \rceil$ point guards. To guard an orthogonal polygon P with h holes, Györi et al. [9] proved that $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are always sufficient to guard P. For survey of art gallery theorems and algorithms, see Ghosh [8], O'Rourke [13], and Urrutia [14].

The minimum guard problem is to find the minimum number of guards that can see every internal point of a polygon. O'Rourke and Supowit [13] showed that the minimum vertex, point and edge guard problems in polygons with holes are NP-hard. Even in the case of polygons without holes, Lee and Lin [12] showed that the minimum vertex, point and edge guard problems are NP-hard. By transforming art gallery problems into set-cover problems, Ghosh [7] first presented approximation algorithms for minimum vertex and edge guard problems for polygons with or without holes. For simple polygons P, approximation algorithms for both problems run in $O(n^4)$ time and yield solutions that can be at most $O(\log n)$ times the optimal solution. For polygons P with holes, approximation algorithms for both problems give the same approximation ratio of $O(\log n)$ but the algorithms take $O(n^5)$ time.

Recently, Efrat and Har-Peled [5] presented randomized approximation algorithms for the minimum vertex guard problem in polygons. For simple polygons P, the randomized approximation algorithm runs in $O(nc_{opt}^2 \log^4 n)$ expected time and the approximation ratio is $O(\log c_{opt})$, where c_{opt} is the number of vertices in the optimal solution. In the worst case, c_{opt} can be a fraction of n.

For polygons P with h holes, the randomized approximation algorithm runs in $O(nhc_{opt}^3 polylog\ n)$ expected time and the approximation ratio is $O(\log n \log(c_{opt} \log n))$. For special classes of polygons, there are approximation algorithms for the minimum point guard problem. Also, there are approximation algorithms (i) for the minimum vertex and point guard problems in 1.5-dimensional terrains, and (ii) for the minimum vertex guard problem in 2.5-dimensional terrains.

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